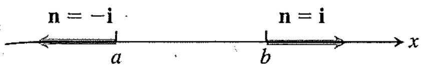



**FIGURE 16.81** The outward unit normals at the boundary of $[a, b]$ in one-dimensional space.

There is still more to be learned here. All these results can be thought of as forms of a *single fundamental theorem*. Think back to the Fundamental Theorem of Calculus in Section 5.4. It says that if $f(x)$ is differentiable on $(a, b)$ and continuous on $[a, b]$, then

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a).$$

If we let $\mathbf{F} = f(x)\mathbf{i}$ throughout $[a, b]$, then $(df/dx) = \nabla \cdot \mathbf{F}$. If we define the unit vector field $\mathbf{n}$ normal to the boundary of $[a, b]$ to be $\mathbf{i}$ at $b$ and $-\mathbf{i}$ at $a$ (Figure 16.81), then

$$\begin{aligned} f(b) - f(a) &= f(b)\mathbf{i} \cdot (\mathbf{i}) + f(a)\mathbf{i} \cdot (-\mathbf{i}) \\ &= \mathbf{F}(b) \cdot \mathbf{n} + \mathbf{F}(a) \cdot \mathbf{n} \\ &= \text{total outward flux of } \mathbf{F} \text{ across the boundary of } [a, b]. \end{aligned}$$

The Fundamental Theorem now says that

$$\mathbf{F}(b) \cdot \mathbf{n} + \mathbf{F}(a) \cdot \mathbf{n} = \int_{[a,b]} \nabla \cdot \mathbf{F}\,dx.$$

The Fundamental Theorem of Calculus, the normal form of Green's Theorem, and the Divergence Theorem all say that the integral of the differential operator $\nabla \cdot$ operating on a field $\mathbf{F}$ over a region equals the sum of the normal field components over the boundary enclosing the region. (Here we are interpreting the line integral in Green's Theorem and the surface integral in the Divergence Theorem as "sums" over the boundary.)

Stokes' Theorem and the tangential form of Green's Theorem say that, when things are properly oriented, the surface integral of the differential operator $\nabla \times$ operating on a field equals the sum of the tangential field components over the boundary of the surface.

The beauty of these interpretations is the observance of a single unifying principle, which we might state as follows.

> **A Unifying Fundamental Theorem of Vector Integral Calculus**
> The integral of a differential operator acting on a field over a region equals the sum of the field components appropriate to the operator over the boundary of the region.

## Exercises 16.8

### Calculating Divergence

In Exercises 1–4, find the divergence of the field.

1. The spin field in Figure 16.12
2. The radial field in Figure 16.11
3. The gravitational field in Figure 16.8 and Exercise 38a in Section 16.3
4. The velocity field in Figure 16.13

### Calculating Flux Using the Divergence Theorem

In Exercises 5–16, use the Divergence Theorem to find the outward flux of $\mathbf{F}$ across the boundary of the region $D$.

5. **Cube** $\mathbf{F} = (y - x)\mathbf{i} + (z - y)\mathbf{j} + (y - x)\mathbf{k}$
   $D$: The cube bounded by the planes $x = \pm 1, y = \pm 1$, and $z = \pm 1$
6. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$
   **a. Cube** $D$: The cube cut from the first octant by the planes $x = 1, y = 1$, and $z = 1$
   **b. Cube** $D$: The cube bounded by the planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$
   **c. Cylindrical can** $D$: The region cut from the solid cylinder $x^2 + y^2 \le 4$ by the planes $z = 0$ and $z = 1$
7. **Cylinder and paraboloid** $\mathbf{F} = y\mathbf{i} + xy\mathbf{j} - z\mathbf{k}$
   $D$: The region inside the solid cylinder $x^2 + y^2 \le 4$ between the plane $z = 0$ and the paraboloid $z = x^2 + y^2$
8. **Sphere** $\mathbf{F} = x^2\mathbf{i} + xz\mathbf{j} + 3z\mathbf{k}$
   $D$: The solid sphere $x^2 + y^2 + z^2 \le 4$
9. **Portion of sphere** $\mathbf{F} = x^2\mathbf{i} - 2xy\mathbf{j} + 3xz\mathbf{k}$
   $D$: The region cut from the first octant by the sphere $x^2 + y^2 + z^2 = 4$
10. **Cylindrical can** $\mathbf{F} = (6x^2 + 2xy)\mathbf{i} + (2y + x^2z)\mathbf{j} + 4x^2y^3\mathbf{k}$
    $D$: The region cut from the first octant by the cylinder $x^2 + y^2 = 4$ and the plane $z = 3$

**11. Wedge** $\mathbf{F} = 2xz\mathbf{i} - xy\mathbf{j} - z^2\mathbf{k}$

$D$: The wedge cut from the first octant by the plane $y + z = 4$ and the elliptical cylinder $4x^2 + y^2 = 16$

**12. Sphere** $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$

$D$: The solid sphere $x^2 + y^2 + z^2 \le a^2$

**13. Thick sphere** $\mathbf{F} = \sqrt{x^2 + y^2 + z^2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

$D$: The region $1 \le x^2 + y^2 + z^2 \le 2$

**14. Thick sphere** $\mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/\sqrt{x^2 + y^2 + z^2}$

$D$: The region $1 \le x^2 + y^2 + z^2 \le 4$

**15. Thick sphere** $\mathbf{F} = (5x^3 + 12xy^2)\mathbf{i} + (y^3 + e^y \sin z)\mathbf{j} + (5z^3 + e^y \cos z)\mathbf{k}$

$D$: The solid region between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 2$

**16. Thick cylinder** $\mathbf{F} = \ln(x^2 + y^2)\mathbf{i} - \left(\frac{2z}{x}\tan^{-1}\frac{y}{x}\right)\mathbf{j} + z\sqrt{x^2 + y^2}\mathbf{k}$

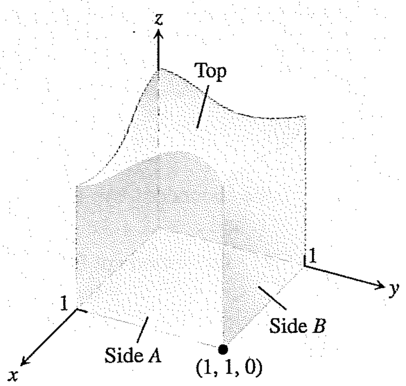
$D$: The thick-walled cylinder $1 \le x^2 + y^2 \le 2, -1 \le z \le 2$

## Theory and Examples

**17. a.** Show that the outward flux of the position vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through a smooth closed surface $S$ is three times the volume of the region enclosed by the surface.

**b.** Let $\mathbf{n}$ be the outward unit normal vector field on $S$. Show that it is not possible for $\mathbf{F}$ to be orthogonal to $\mathbf{n}$ at every point of $S$.

**18.** The base of the closed cubelike surface shown here is the unit square in the $xy$-plane. The four sides lie in the planes $x = 0$, $x = 1$, $y = 0$, and $y = 1$. The top is an arbitrary smooth surface whose identity is unknown. Let $\mathbf{F} = x\mathbf{i} - 2y\mathbf{j} + (z + 3)\mathbf{k}$ and suppose the outward flux of $\mathbf{F}$ through Side $A$ is 1 and through Side $B$ is $-3$. Can you conclude anything about the outward flux through the top? Give reasons for your answer.




**19.** Let $\mathbf{F} = (y \cos 2x)\mathbf{i} + (y^2 \sin 2x)\mathbf{j} + (x^2y + z)\mathbf{k}$. Is there a vector field $\mathbf{A}$ such that $\mathbf{F} = \nabla \times \mathbf{A}$? Explain your answer.

**20. Outward flux of a gradient field** Let $S$ be the surface of the portion of the solid sphere $x^2 + y^2 + z^2 \le a^2$ that lies in the first octant and let $f(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2}$. Calculate

$$\iint_S \nabla f \cdot \mathbf{n}\, d\sigma.$$

($\nabla f \cdot \mathbf{n}$ is the derivative of $f$ in the direction of outward normal $\mathbf{n}$.)

**21.** Let $\mathbf{F}$ be a field whose components have continuous first partial derivatives throughout a portion of space containing a region $D$ bounded by a smooth closed surface $S$. If $|\mathbf{F}| \le 1$, can any bound be placed on the size of

$$\iiint_D \nabla \cdot \mathbf{F}\, dV?$$

Give reasons for your answer.

**22. Maximum flux** Among all rectangular solids defined by the inequalities $0 \le x \le a, 0 \le y \le b, 0 \le z \le 1$, find the one for which the total flux of $\mathbf{F} = (-x^2 - 4xy)\mathbf{i} - 6yz\mathbf{j} + 12z\mathbf{k}$ outward through the six sides is greatest. What *is* the greatest flux?

**23.** Calculate the net outward flux of the vector field

$$\mathbf{F} = xy\mathbf{i} + (\sin xz + y^2)\mathbf{j} + (e^{xy^2} + x)\mathbf{k}$$

over the surface $S$ surrounding the region $D$ bounded by the planes $y = 0, z = 0, z = 2 - y$ and the parabolic cylinder $z = 1 - x^2$.

**24.** Compute the net outward flux of the vector field $\mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/(x^2 + y^2 + z^2)^{3/2}$ across the ellipsoid $9x^2 + 4y^2 + 6z^2 = 36$.

**25.** Let $\mathbf{F}$ be a differentiable vector field and let $g(x, y, z)$ be a differentiable scalar function. Verify the following identities.

**a.** $\nabla \cdot (g\mathbf{F}) = g\nabla \cdot \mathbf{F} + \nabla g \cdot \mathbf{F}$

**b.** $\nabla \times (g\mathbf{F}) = g\nabla \times \mathbf{F} + \nabla g \times \mathbf{F}$

**26.** Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be differentiable vector fields and let $a$ and $b$ be arbitrary real constants. Verify the following identities.

**a.** $\nabla \cdot (a\mathbf{F}_1 + b\mathbf{F}_2) = a\nabla \cdot \mathbf{F}_1 + b\nabla \cdot \mathbf{F}_2$

**b.** $\nabla \times (a\mathbf{F}_1 + b\mathbf{F}_2) = a\nabla \times \mathbf{F}_1 + b\nabla \times \mathbf{F}_2$

**c.** $\nabla \cdot (\mathbf{F}_1 \times \mathbf{F}_2) = \mathbf{F}_2 \cdot \nabla \times \mathbf{F}_1 - \mathbf{F}_1 \cdot \nabla \times \mathbf{F}_2$

**27.** If $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a differentiable vector field, we define the notation $\mathbf{F} \cdot \nabla$ to mean

$$M\frac{\partial}{\partial x} + N\frac{\partial}{\partial y} + P\frac{\partial}{\partial z}.$$

For differentiable vector fields $\mathbf{F}_1$ and $\mathbf{F}_2$, verify the following identities.

**a.** $\nabla \times (\mathbf{F}_1 \times \mathbf{F}_2) = (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 - (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\nabla \cdot \mathbf{F}_2)\mathbf{F}_1 - (\nabla \cdot \mathbf{F}_1)\mathbf{F}_2$

**b.** $\nabla(\mathbf{F}_1 \cdot \mathbf{F}_2) = (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 + \mathbf{F}_1 \times (\nabla \times \mathbf{F}_2) + \mathbf{F}_2 \times (\nabla \times \mathbf{F}_1)$

**28. Harmonic functions** A function $f(x, y, z)$ is said to be *harmonic* in a region $D$ in space if it satisfies the Laplace equation

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

throughout $D$.

**a.** Suppose that $f$ is harmonic throughout a bounded region $D$ enclosed by a smooth surface $S$ and that $\mathbf{n}$ is the chosen unit normal vector on $S$. Show that the integral over $S$ of $\nabla f \cdot \mathbf{n}$, the derivative of $f$ in the direction of $\mathbf{n}$, is zero.

**b.** Show that if $f$ is harmonic on $D$, then

$$\iint_S f\nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D |\nabla f|^2\, dV.$$